E-ISSN: 2707-8299
P-ISSN: 2707-8280
IJSDE 2024; 5(1): 01-06
Received: 01-11-2023
Accepted: 06-12-2023

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Buckling analysis of isotropic Euler-Bernoulli beam resting on Winkler elastic foundation

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Abstract
Buckling analysis of isotropic beam based on Euler-Bernoulli beam theory EBT resting on elastic foundation is investigated in this paper. The isotropic beam is exposed to an axial compressive force. Using potential energy principle the governing equations are obtained. The critical buckling load was evaluated using Navier's method. Numerical analysis show effect both spring constant with longitudinal wave number on critical buckling loads. Validation of the numerical findings of this paper indicates that comparing the results obtainable in the previous study shows excellent agreement. The numerical results show that critical buckling load increasing with increases spring constant.

Keywords: Buckling analysis, isotropic beam, Euler-Bernoulli beam, elastic foundation

Introduction
The isotropic beam is materials that have the same properties and the modulus of elasticity in all points or in all directions. Isotropic beams, a popular structural feature for the applications of aeronautics and astronautics. For many practical engineering applications, metals can be considered as globally isotropic materials. J.N. Reddy [1] investigated equations of motion by Hamilton principle and using different beam theories. Analytical solutions are used to determine each critical buckling load, natural frequency, and maximum deflection. Soltani and Mohammadi [2] derived the governing equations using Euler-Bernoulli beam theory. To evaluate the critical buckling, the Differential Quadrature Method (DQM) was used for simply supported. Karamanli [3] studied the governing equations for simply supported and cantilever beams with different boundary conditions by using Timoshenko, and Euler-Bernoulli beam theories subjected to uniformly distributed load. The SSPH method is used to determine the maximum deflection by solving the governing equations. Zhu *et al*. [4] studied the buckling problem using Erin- gen's two-phase nonlocal integral model. The findings of the numerical approach used are in good agreement with results literature available. Salehi and Ansari [5] examined the linear viscoelastic buckling based on Timoshenko beam and Euler-Bernoulli theories. The transvers deflection and critical buckling are determined by solving the governing equations for a beam subjected to axial and transverse loads for various boundary conditions. The numerical method used to solve the governing equations demonstrates that its results are in strong accordance with the literature's findings. Shimpi *et al*. [6] studied the governing equations based on Hamilton's principle and using refined theory. The numerical analysis using to determine the maximum deflection and natural frequency. Elshafei [7] derived equations of motion by Hamilton's principle. For isotropic and orthotropic beams exposed to distributed transverse loads and using varying boundary conditions, the finite element approach is used to find both maximal deflection and natural frequency. Saba and Mangulkar [8] studied the governing equations using the total potential energy principle based on hyperbolic shear deformation theory for isotropic beam subjected to uniformly varying load with cantilever beam boundary conditions. Through numerical analysis, the governing equations are solved to obtain the maximum transverse deflection. Magnucki *et al*. [9] studied an isotropic beam subjected to three-point bending to find the governing equilibrium equations using the shear deformation theory under the total potential energy principle. The finite element method using for calculate maximal deflection, and we found it in good agreement when contrasting its observations with the results of numerical methods available in the literature. Sohani and Eipakchi [10] derived the governing equations using Timoshenko beam and Euler-Bernoulli theories. Numerical results, which contained the numerical results available in literature,

were in a good agreement with them. Onyia and Lato [11] investigated the governing equations by applying Timoshenko beam finite element for isotropic beam exposed to uniformly distributed load with a point load. By solving the governing equations, the maximum deflection can be obtained for different boundary conditions. The numerical findings are consistent with the numerical results available in the previous study. Armagan Karamanli [12] derived the governing equations by using EBT and Timoshenko beam theories and based on the total potential energy principle for isotropic beam exposed to a uniformly distributed load. The maximal transverse deflections and axial stresses for the various boundary conditions are calculated using SSPH method. OSADEBE *et al*. [13] investigated the governing equations by using Timoshenko beam theory for isotropic beam subjected to a point load. The maximum deflection and natural frequencies are determine by using finite element method. Wang *et al*. [14] studied the governing equations by Timoshenko beam for isotropic beam. Yesilce and Catal [15] examined governing equations of motion using Hamilton principle based on Timoshenko, Euler-Bernoulli, and Reddy-Bickford beams theories for isotropic beam subjected to axial compressive force resting on elastic foundation. The natural frequencies are found using the differential transform method under various boundary conditions. Wang *et al*. [16] presented vibration, buckling, and bending analysis of isotropic beam. For various boundary conditions, the finite element approach is adopted to find the transvers deflection, critical buckling, and natural frequencies. Soares *et al*. [17] derived the governing equations using isotropic Timoshenko beam theory resting on Pasternak foundation. The transvers deflection and the natural frequency of simply supported are calculated using the finite element method. They were found to be in outstanding agreement with the findings obtainable in the previous study when analyzing the numerical results. Mirzabeigy *et al*. [18] studied free vibration and buckling analysis for isotropic beam subjected to axial force. Under varying boundary conditions, the numerical results indicate both natural frequencies and critical buckling. FENG *et al*. [19] derived governing equations using Timoshenko beam theory. The finite element under varying boundary conditions using to calculate both maximal deflection and natural frequency. Carrer *et al*. [20] examined the static analysis of the isotropic beam exposed to a uniform distributed load. Finite element method using to determine the maximum deflection.

Governing Equations and Formulation

According to Fig. 1, consider a schematic view of the isotropic beam of thickness h , Length L, and width *b*. The beam is exposed to axial force N_{xo} resting on elastic foundation and spring constant k_w . Young's modulus E of the beam belongs to steel.

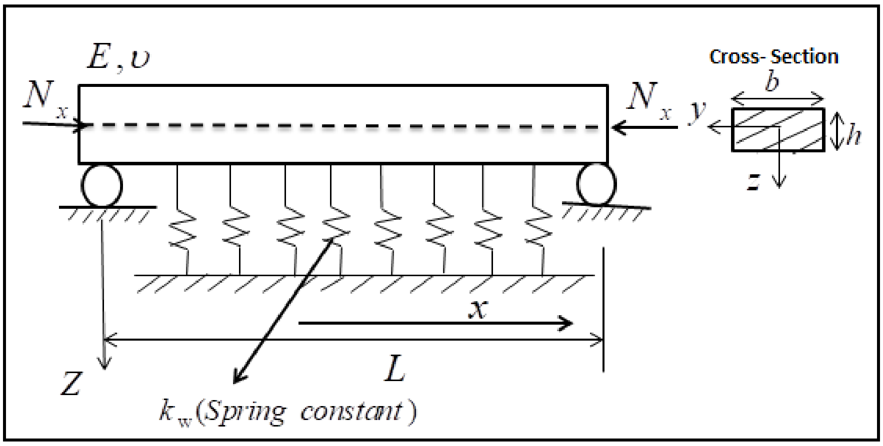

Fig 1: Isotropic beam subjected to axial compression load resting on elastic foundation

According to EBT the displacement field as:

$$u(x,y,z) = u_0(\mathrm{x}) - z\frac{\partial w_0}{\partial x} \qquad (1)$$

$$v(x,y,z) = 0 \qquad (2)$$

$$w(x,\mathrm{y},\mathrm{z}) = w_0(x) \qquad (3)$$

Assumed u, v, w is the displacement components of x, y, and z, respectively and u_0 & w_0 are the axial and transverse displacements in x-z directions and v the displacement of the beam in the y-direction, and $\mathrm{w}_0(x)$ transvers deflection.

From Eq. (1) can find the normal strain following as:

$$\varepsilon_{xx} = \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w}{\partial x^2} \qquad (4)$$

The strain energy (potential energy) it is possible to write the following:

$$\mathrm{U} = \frac{1}{2}\int_v \sigma_{ij}\varepsilon_{ij}dV \qquad (5a)$$

$$\delta\mathrm{U} = \int_v \sigma_{ij}\delta\varepsilon_{ij}dV \qquad (5b)$$

$$\delta\mathrm{U} = \int_V \sigma_{xx}\delta\varepsilon_{xx} + \sigma_{yy}\delta\varepsilon_{yy} + \sigma_{zz}\delta\varepsilon_{zz} + \sigma_{xy}\delta\varepsilon_{xy} + \sigma_{yz}\delta\varepsilon_{yz} + \sigma_{xz}\delta\varepsilon_{xz} + \sigma_{yx}\delta\varepsilon_{yx} + \sigma_{zy}\delta\varepsilon_{zy} + \sigma_{zx}\delta\varepsilon_{zx}dV \qquad (5c)$$

Where:

$$\varepsilon_{yy} = 0 \ , \varepsilon_{zz} = 0 \qquad (6a)$$

$$\gamma_{xy} = \gamma_{yx} = 0 \ , \ \gamma_{xz} = \gamma_{zx} = 0 \ , \ \gamma_{yz} = \gamma_{zy} = 0 \qquad (6b)$$

From Eq. (5c) and using Eqs. (6a) & (6b) can be written strain energy (potential energy) of the isotropic EBT as follows:

$$\delta U = \int_0^L \int_A (\sigma_{xx}\, \delta\varepsilon_{xx})\, dA\, dx \qquad (7)$$

Where σ_{xx} , ε_{xx} are axial stress, axial strain in "x" direction and δ is the variational operator.

Substituting Eq. (4) into Eq. (7) and using Eq. (9) can be written the final strain energy (potential energy) for isotropic EBT as follows:

$$\delta U = \int_0^L \left[\left(-\frac{\partial N_x}{\partial x}\, \delta u_0 - \frac{\partial^2 M_x}{\partial x^2} \right) \delta w_0 \right] dx \qquad (8)$$

Where:

$$N_x = \int_A \sigma_{xx}\, dA \quad , \quad M_x = \int_A \sigma_{xx}\, z\, dA \qquad (9)$$

N_x , M_x are axial force and bending moment respectively.

External work for isotropic EBT is considered as follows:

$$W_{ext} = \frac{-1}{2} \int_A (F_{ext} w_0)\, dA \qquad (10a)$$

$$\delta W_{ext} = -b \int_0^L (F_{ext}\, \delta w_0)\, dx \qquad (10b)$$

$$F_{ext} = F_{elastic\ foundation} + F_{buckling} \left(\frac{\partial^2 w}{\partial x^2} \right) \qquad (10c)$$

$$F_{elastic\ foundation} = -k_w w \ , F_{buckling} = N_{x0} \frac{\partial^2 w}{\partial x^2} \qquad (10d)$$

Substituting Eqs. (10c) & (10d) into Eq. (10b) can be written final external work for isotropic EBT as:

$$\delta W_{ext} = -b \int_0^L \left[\left(-k_w w_0 + N_{x0} \frac{\partial^2 w}{\partial x^2} \right) \delta w_0 \right] dx \qquad (11)$$

The governing equations for the isotropic EBT can be derived applying the following using the entire potential energy principle:

$$\Pi = (U + W_{ext}) \qquad (12a)$$

$$\delta \Pi = 0 \qquad (12b)$$

$$\delta U + \delta W_{ext} = 0 \qquad (12c)$$

Substituting Eqs. (8) and (11) into Eq. (12c) and setting the coefficients of δu_0 & δw_0 to zero the can determine the final equations of equilibrium of isotropic EBT as follows.

$$\delta u_0 : \ -\frac{\partial N_x}{\partial x} = 0 \qquad (13a)$$

$$\delta w_0 : \ -\frac{\partial^2 M_x}{\partial x^2} + b\, k_w w_0 - b\, N_{x0} \frac{\partial^2 w}{\partial x^2} = 0 \qquad (13b)$$

By using Hooke's law

$$\sigma_{xx} = E(z) * \varepsilon_{xx} \qquad (14)$$

Substituting Equation (14) into equation (9). it is possible to write the axial normal force and bending moment for isotropic EBT as follows:

$$N_x = \int_{-h/2}^{h/2} E\, \varepsilon_{xx}\, b\, dz \qquad (15a)$$

$$M_x = \int_{-h/2}^{h/2} E\, z\, \varepsilon_{xx}\, b\, dz \qquad (15b)$$

Substituting Eq. (4) into Eqs. (15a) & (15b) we have:

$$N_x = Ebh \frac{\partial u_0}{\partial x} \qquad (16a)$$

$$M_x = \frac{-Ebh^3}{12} \frac{\partial^2 w}{\partial x^2} \qquad (16b)$$

Substituting Eqs. (16a) & (16b) into Eqs. (13a) & (13b) can be determine the final governing equations of equilibrium for isotropic EBT as follows:

$$\frac{\partial^2 u_0}{\partial x^2} = 0 \qquad (17a)$$

$$-\frac{Eh^3}{12} \frac{\partial^4 w}{\partial x^4} - k_w w_0 + N_{x0} \frac{\partial^2 w}{\partial x^2} = 0 \qquad (17b)$$

Analytical Solution of Buckling for Isotropic Beam

Based on the Euler-Bernoulli beam theorem, the governing equation of the isotropic beam subjected to axial compressive force resting on the Winkler elastic base can be solved. The Navier-type solution method is used to calculate the critical buckling load of the isotropic beam.

$$u_{(x)} = \sum_{m=1,2,3}^{\infty} U_m \cos\left(\frac{m\pi x}{L} \right) \qquad (18a)$$

$$w_{(x)} = \sum_{m=1,2,3}^{\infty} W_m \sin\left(\frac{m\pi x}{L} \right) \qquad (18b)$$

According Navier-type solution method can determine the critical buckling load for isotropic beam and the variables $u_{(x)}$, $w_{(x)}$ are described as below:

$$\left[-\left(\frac{m\pi}{L} \right)^2 \right] U_m = 0 \qquad (19a)$$

$$\left[-\frac{Eh^3}{12} \left(\frac{m\pi}{L} \right)^4 - N_{x0} \left(\frac{m\pi}{L} \right)^2 - k_w \right] W_m = 0 \qquad (19b)$$

Where U_m, W_m are the unknown Fourier coefficients. Substituting Eqs. (18a) & (18b) into Eqs. (17a) &(17b) the governing equations given by Eqs. (19a)&(19b) can get the following:

$$\begin{bmatrix} \left(\frac{m\pi}{L}\right)^2 & 0 \\ 0 & \frac{Eh^3}{12}\left(\frac{m\pi}{L}\right)^4 + N_{x1}\left(\frac{m\pi}{L}\right)^2 + k_w \end{bmatrix} \begin{bmatrix} U_m \\ W_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{20}$$

Using Eqs. (19a) & (19b) the final matrix form of isotropic EBT can be written as follows:

Numerical Results

Through numerical results can the critical buckling load is calculated for isotropic beam simply supported boundary conditions centered on EBT and using Navier's-type solution method. The beam subjected to axial compressive force. Assuming that the Length of the isotropic beam is L=1m and thickness h=0.1m and width b=0.1m also, the longitudinal wavenumber m=1.

Dimensionless critical buckling load is defined as:

$$N_{cr} = \frac{N * L^2}{E * I} \qquad (21)$$

Table 1. shows the non-dimensionless critical buckling P_{cr} versus values slenderness ratio L/h using the potential energy principle and using Navier's method. It shows that the data of the non-dimensionless critical buckling load are in an outstanding contract with the solution of Reddy [1]. Shown from this table that the values of the critical buckling P_{cr} are constant with different three values of the slenderness ratio.

Table 1: The non-dimensionless non-dimensional critical buckling load $N_{cr} = N * L^2 / E * I$ of isotropic EBT with values of the slenderness ratio L/h

Slenderness ratio(L/h)	non-dimensionless non-dimensional critical buckling load	
	Work EBT	Present Reddy [1]
L/h=100	9.8696	9.8696
L/h=20	9.8696	9.8696
L/h=10	9.8696	9.8696

Figures (2 & 3) show the variation of the critical buckling load P_{cr} of an isotropic beam simply supported boundary conditions with slenderness ratio for different values of the k_w centered on EBT and adopting the total potential energy principle. Figures (2 & 3) show that the critical buckling increases as increasing in the spring constant because with the increase in the values of the spring constant the beam becomes more stiffer. It can also be shown from Figures (2 & 3) that the critical buckling declines as the slenderness ratio values increase.

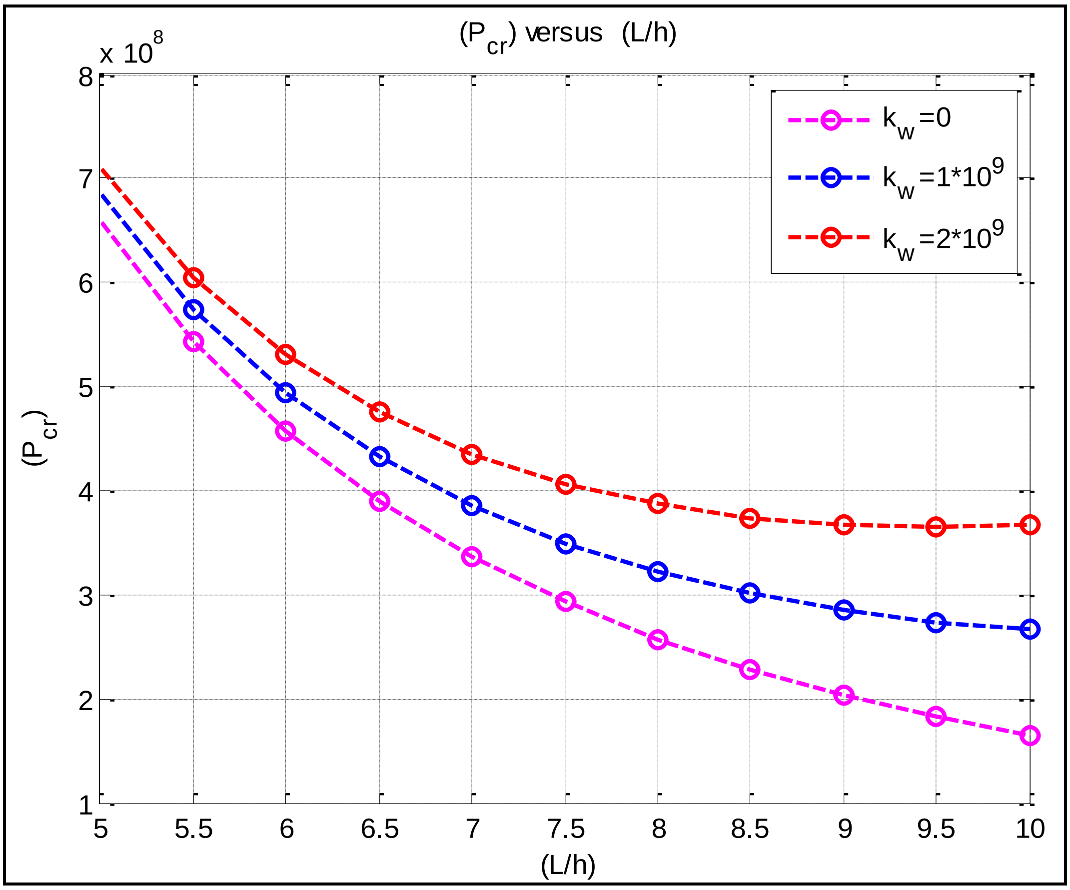

Fig 2: Spring constant with critical buckling for isotropic beam versus slenderness ratio

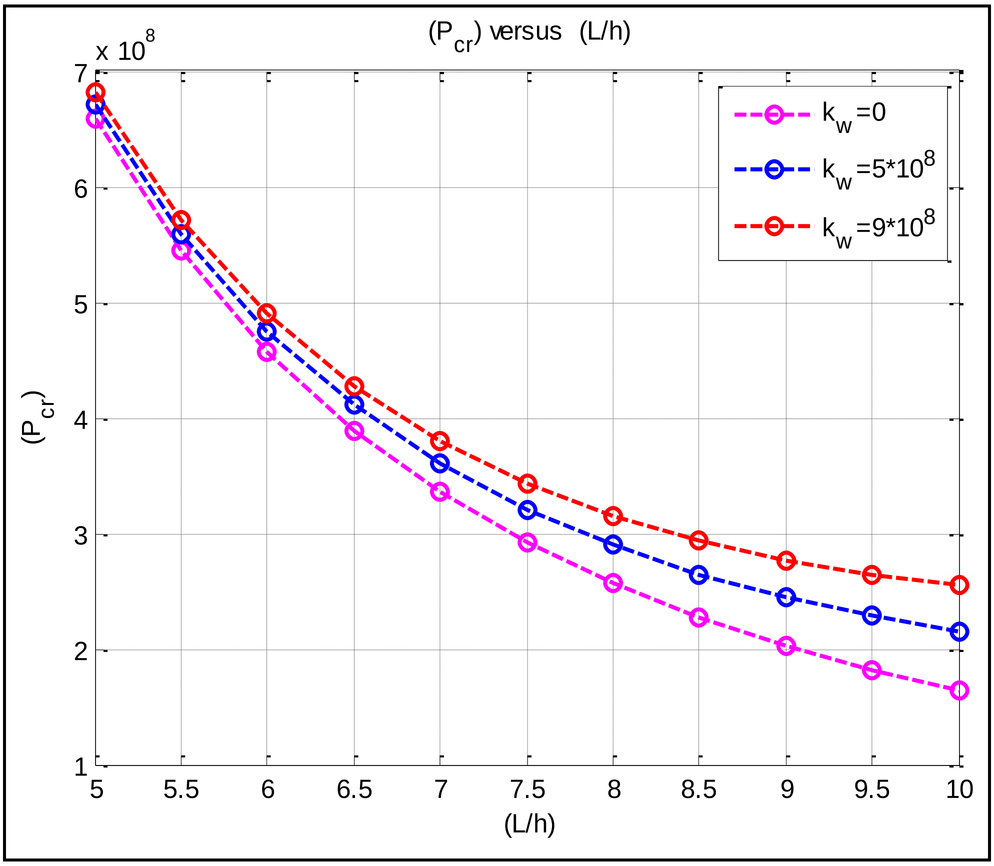

Fig 3: Spring constant with critical buckling for isotropic beam and slenderness ratio

Figure 4 indicates the variance of the critical buckling of the isotropic beam that simply supports boundary conditions versus the slenderness ratio based on the EBT for five distinct longitudinal m wave number m values. It can be seen from this fig.4 that with an increase in the longitudinal wave number m, the critical buckling P_{cr} enhances. From this figure also, the critical buckling decreases with increase in the values of the slenderness ratio.

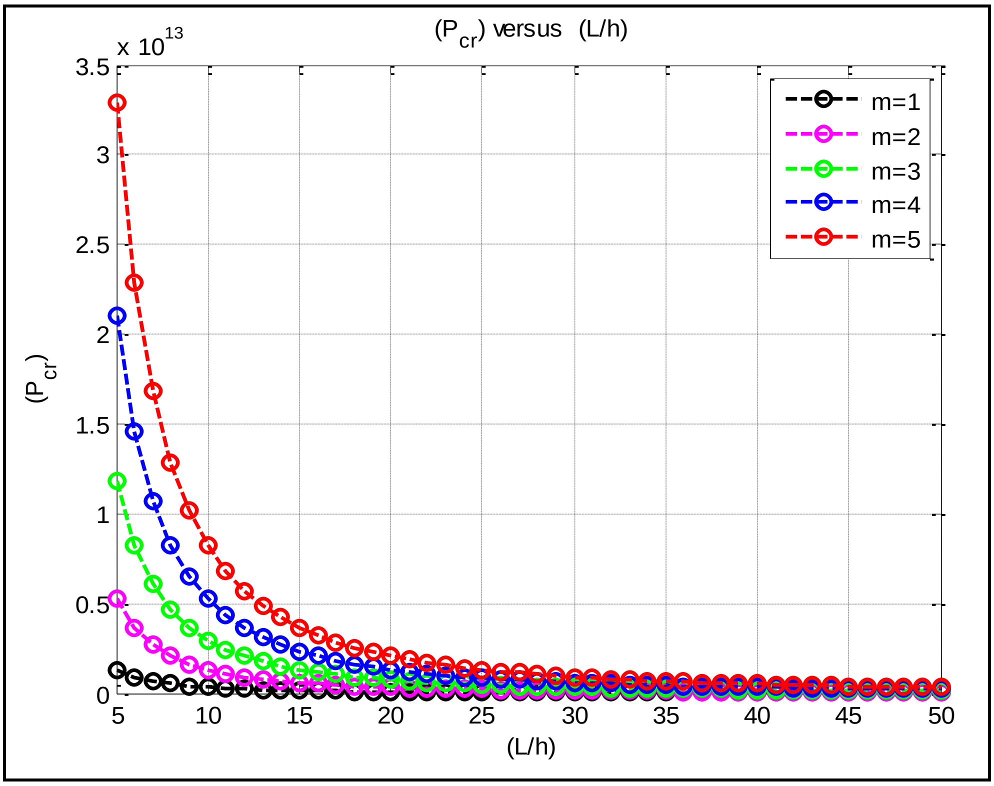

Fig 4: Effect for m on P_{cr} of isotropic Euler-Bernoulli beam theory EBT with L/h

Conclusion

Buckling analysis of isotropic beam based on Euler-Bernoulli beam theory EBT resting on elastic foundation is investigated in this paper. The isotropic beam is exposed to an axial compressive force. Using potential energy principle the governing equations are obtained. The critical buckling load was evaluated using Navier's method. Numerical analysis show effect both spring constant with longitudinal wave number on critical buckling loads. Validation of the numerical findings of this paper indicates that comparing the results obtainable in the previous study shows excellent agreement.

1. Critical buckling load decreases as increasing in the values of the slenderness ratio.
2. With increasing in the values of the spring constant the critical buckling increases because with increase in the values of the spring constant the beam becomes stiffer.
3. Critical buckling load increases as increasing in the values of the longitudinal wave number of the simply supported isotropic Euler-Bernoulli beam.

References

1. Reddy JN. Nonlocal theories for bending, buckling and vibration of beams. Int J Eng Sci. 2007;45:288-307.
2. Soltani M, Mohammadi M. Stability Analysis of Non-Local Euler-Bernoulli Beam with Exponentially Varying Cross-Section Resting on Winkler-Pasternak Foundation. Numer Methods Civil Eng. 2018;2:67-77.
3. Karamanli A. Flexure Analysis of Laminated Composite and Sandwich Beams Using Timoshenko Beam Theory. J Polytech. 2018;21(3):633-643.
4. Zhu X, Wang Y, Dai HH. Buckling analysis of Euler-Bernoulli beams using Eringen's two-phase nonlocal model. Int J Eng Sci. 2017;116:130-140.
5. Salehi M, Ansari F. Viscoelastic Buckling of Euler-Bernoulli and Timoshenko Beams under Time Variant General Loading Conditions. Iran Polym J. 2006;15:184-193.
6. Shimpi RP, Guruprasad PJ, Pakhare KS. Simple Two Variable Refined Theory for Shear Deformable Isotropic Rectangular Beams. J Appl Comput Mech. 2020;6:394-415.
7. Elshafei MA. FE Modeling and Analysis of Isotropic and Orthotropic Beams Using First Order Shear Deformation Theory. Mater Sci Appl. 2013;4:77-102.
8. Saba SN, Mangulkar MN. Bending Analysis of Isotropic Beam using "Hyperbolic Shear Deformation Theory. IOSR J Mech Civil Eng. 2018;15:31-39.
9. Magnucki K, Lewinski J, Magnucka-Blandzi E. An improved shear deformation theory for bending beams with symmetrically varying mechanical properties in the depth direction. Springer. 2020;231:4381-4395.
10. Sohani F, Eipakchi HR. Analytical solution for modal analysis of Euler-Bernoulli and Timoshenko beam with an arbitrary varying cross-section. Math Models Eng. 2018;4:164-174.
11. Onyia ME, Rowland-Lato EO. Finite Element Analysis of Timoshenko Beam Using Energy Separation Principle. Int J Eng Res Technol. 2020;13:28-35.
12. Karamanli A. Analysis of isotropic tapered beams by using symmetric smoothed particle hydrodynamics method. New Trends Math Sci. 2016;4:145-162.
13. Osadebe NN, Agunwamba JC, Onyia ME, Rowland-Lato EO. Free Vibration Analysis of Timoshenko Beam Using Energy Separation Principle. Int J Sci Eng Res. 2018;9:340-352.
14. Li XY, Wang XH, Chen YY, Tan Y, Cao HJ. Bending, Buckling and Free Vibration of an Axially Loaded Timoshenko Beam with Transition Parameter: Direction of Axial Force. Int J Mech Sci. 2020;176:1-38.
15. Yesilce Y, Catal HH. Solution of free vibration equations of semi-rigid connected Reddy-Bickford beams resting on elastic soil using the differential transform method. Springer. 2011;81:199-213.
16. Wang CM, Lam KY, He XQ. Exact Solutions for Timoshenko Beams on Elastic Foundations Using Green's Functions. Mech Struct Mach. 1998;26:101-113.
17. Soares LS, Bezerra WKS, Hoefel SDS. Dynamic Analysis Of Timoshenko Beam On Pasternak Foundation. Am Congress Comput Methods Eng. 2017:1-15.
18. Mirzabeigy A, Haghpanahi M, Madoliat R. A New Finite Element Formulation for Buckling and Free Vibration Analysis of Timoshenko Beams on Variable Elastic Foundation. J Solid Mech. 2017;9:276-290.
19. Feng H, Cui X, Li G. Static And Dynamic Analysis Of Timoshenko Beam Using Nodal Integration Technique. Int J Appl Mech. 2012;4:1-20.
20. Carrer JAM, Mansur WJ, Sciuciato RF, Fleischfresser SA. Analysis Of Euler-Bernoulli And Timoshenko Beams By The Boundary Element Method. J Automotive Appl Mech. 2014;1:1-17.